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FROM MIXTURE THEORY TO BIOT'S APPROACH FOR POROUS MEDIA

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Abstract—Two apparently different approaches are used in dealing with the mechanics of a deformable porous medium: mixture theories on the one hand, and purely macroscale theories, which are mainly associated with the work of Biot, on the other hand. In the mixture theories, the porous medium is represented by spatially superposed interacting media, while macroscale theories assume that standard concepts of continuum mechanics are still relevant at the macro-level. The aim of this paper is two-fold. First, it is shown that the macroscale field equations derived from mixture theories can be reformulated in terms of the measurable quantities involved in the macroscale theories. Second, it is demonstrated how these field equations, including the fundamental inequality obtained from the second law, entail the existence of a macroscale Ψ-potential upon which a thermodynamically consistent formulation of the constitutive equations can be firmly founded. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

Two apparently different approaches are used in dealing with the mechanics of a deformable porous medium: mixture theories on the one hand, and purely macroscale theories, which are mainly associated with the work of Biot, on the other hand. In the mixture theories, the porous medium is represented by spatially superposed interacting media, while macroscale theories assume that standard concepts of continuum mechanics are still relevant at the macro-level.

In the mixtures theories, the relevance of macroscale concepts, such as the stress tensor, are proved. The field equations are properly derived through averaging procedures, generally using an Eulerian description in order to preserve the symmetry of the role played by each constituent. These theories are well suited to take various phenomena into account, including complex interactions between constituents. However, when involving a deformable solid matrix, the constitutive equations in these theories are to a certain extent introduced in an *ad hoc* way, in order to allow their experimental identification in the laboratory at the macro-level.

In contrast to mixture theories, the macroscale theories assume from the onset that the stress and other related concepts hold at the macro-level, as well as most of the classical field equations of continuum mechanics. In these theories, the constitutive equations involve well defined and measurable quantities at the macro-level, as for instance the variation in fluid mass content. They are generally formulated in a Lagrangian description with the help of a macroscale potential whose existence implies very useful symmetry properties.

The aim of this paper is two-fold. First, we show that the macroscale field equations derived from mixture theories can be reformulated in terms of the measurable quantities

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involved in the macroscale theories. Second, we demonstrate how these field equations, including the fundamental inequality obtained from the second law, entail the existence of a macroscale Ψ-potential upon which a thermodynamically consistent formulation of the constitutive equations can be firmly founded. It is hoped that the links established in this paper will provide an incentive to rely with more confidence on macroscale theories, which often turn out to be more tractable for engineering applications.

Since it is the connection with the macroscale theories which is of main concern here, the mathematical averaging procedures of mixture theories will only be considered briefly in what follows. For an extensive presentation the reader can refer to the abundant existing literature, in particular to Bowen (1976, 1982), Bedford and Drumheller (1983), Truesdell (1984), Bear and Bachmat (1990), Morland (1992), de Boer (1996). Neither the bases of the macroscale theories will be detailed. The interested reader can refer to the pioneering work of Biot (1941, 1956, 1972, 1977), which has been placed in the more general context of irreversible thermodynamics of polyphasic reactive open continua by Coussy (1995), and extended to non-ideal mixtures by Dormieux *et al.* (1995). Owing also to the purpose of this paper, we will restrict considerations to the case of a porous medium saturated by only one fluid, with no sources of any kind nor chemical reaction occurring between the solid matrix and the fluid. Nevertheless the approach will be developed in the general context of finite deformations to show that some results are not subject to linearization procedures of any kind, in particular of geometrical nature.

2. PHYSICAL BALANCE LAWS IN MIXTURE THEORIES

2.1. *Two different geometric scales*

Considering the description of the porous medium as a continuum, we have to distinguish two geometric scales. At the called microscale, each constituent occupies a specific domain, while at the macroscale, a skeleton particle and a fluid particle are superposed at the same given geometrical point. The skeleton particle is formed of the solid matrix and of the porous connected space, while the fluid particle is formed of the fluid saturating the connected space and the remaining space. The current porous volume fraction or porosity ϕ is such that the porous connected space through which fluid flow occurs within the elementary volume $\mathrm{d}\Omega_t$ (occupied by both particles) is $\phi\,\mathrm{d}\Omega_t$, while the solid matrix occupies the remaining volume $(1-\phi)\,\mathrm{d}\Omega_t$. Finally, a saturated porous medium is represented as the superposition of two interacting continua, the skeleton and the fluid, having distinct kinematics. For the sake of simplicity, we assume that no mass exchange takes place between these two constituents of the porous medium.

2.2. *Averaging procedure*

Aiming at establishing the equations which express the physical balance laws at the macroscale from their expressions at the microscale, an averaging procedure has to be selected. The technique used here is the spatial convolution method (Marle, 1982; Gilbert, 1987).

First we distinguish between a point at the microscale, located by its position vector $\mathbf{z}$, and a point at the macroscale, identified by its position vector $\mathbf{x}$. We then introduce a sphere $D(O)$ or radius R centred at the origin O and a sphere $D(\mathbf{x})$, also of radius R, centred at point $\mathbf{x}$. The radius R is taken to be large with respect to the characteristic length of the microstructure. It is assumed that $D(\mathbf{x})$ can be considered as a representative elementary volume (Bear and Bachmat, 1990) and is thus identified to the elementary volume $\mathrm{d}\Omega_t$. Let $f(\mathbf{z})$ be a function defined at the microscale in $D(O)$, such that it satisfies

$$\int_{D(O)} f(\mathbf{z})\,\mathrm{d}V = 1 \tag{1}$$

With these notations, the apparent mean value $\langle\psi_{(\alpha)}\rangle^{\alpha}(\mathbf{x},t)$ of the function $\psi_{(\alpha)}(\mathbf{z},t)$ over the constituent α contained in $D(\mathbf{x})$ is defined as

$$\langle \psi_{(\alpha)} \rangle^{\alpha}(\mathbf{x}, t) = \int_{D(\mathbf{x})} f(\mathbf{x}-\mathbf{z})\psi_{(\alpha)}(\mathbf{z}, t) I_{(\alpha)}(\mathbf{z}, t)\,\mathrm{d}V \tag{2}$$

where $I_{(\alpha)}(\mathbf{z}, t)$ is the indicator function of the constituent α. For instance, the volume fraction ϕ^{α} of constituent α is equal to $\langle I_{(\alpha)} \rangle^{\alpha}$ and the relative density $\rho_a^{\alpha}(\mathbf{x}, t)$ of constituent α at the macroscale is related to the density $\rho_{(\alpha)}(\mathbf{z}, t)$ at the microscale by the relationship $\rho_a^{\alpha} = \langle \rho_{(\alpha)} \rangle^{\alpha}$. In the following, we shall also use the intrinsic density at the macroscale $\rho^{\alpha} = \rho_a^{\alpha}/\phi^{\alpha}$ which refers to the volume actually occupied by the constituent α at the microscale.

A natural choice for f would be the function which is zero outside $D(O)$ and which is constant and equal to the inverse of the volume of $D(O)$ inside $D(O)$. However, insofar as we have to determine the derivatives of $\langle \psi_{(\alpha)} \rangle^{\alpha}$ with respect to time t and to the position vector $\mathbf{x}$, it will be easier to work with a smooth weighing function. More precisely, the derivation of the balance laws at the macroscale from their expressions at the microscale relies on the following results (Estrada and Kanwal, 1980)

$$\mathrm{grad}_{\mathbf{x}} \langle \psi_{(\alpha)} \rangle^{\alpha} = \langle \mathrm{grad}_{\mathbf{z}}\, \psi_{(\alpha)} \rangle^{\alpha} - \int_{\Gamma^{\alpha}} f(\mathbf{x}-\mathbf{z})\psi_{(\alpha)}(\mathbf{z}, t) \otimes \mathbf{n}_{(\alpha)}(\mathbf{z}, t)\,\mathrm{d}S \tag{3}$$

$$\frac{\partial}{\partial t}\langle \psi_{(\alpha)} \rangle^{\alpha} = \left\langle \frac{\partial}{\partial t}\psi_{(\alpha)} \right\rangle^{\alpha} + \int_{\Gamma^{\alpha}} f(\mathbf{x}-\mathbf{z})\psi_{(\alpha)}(\mathbf{z}, t)\mathbf{v}_{(\alpha)}(\mathbf{z}, t) \cdot \mathbf{n}_{(\alpha)}(\mathbf{z}, t)\,\mathrm{d}S \tag{4}$$

where Γ^{α} is the boundary of the domain occupied by the constituent α inside of $D(\mathbf{x})$, $\mathbf{n}_{(\alpha)}(\mathbf{z}, t)$ is the outwards unit normal and $\mathbf{v}_{(\alpha)}(\mathbf{z}, t)$ designates the Eulerian velocity of constituent α at the microscale. Note that in (3) and (4), $\psi_{(\alpha)}$ is assumed to be a continuous vectorial or scalar function over the domain occupied by the constituent α.

2.3. *Deformation and kinematics at the macroscale, Eulerian approach*

Since the symmetry of the role played by the different constituents is to be preserved in the mixture theory, the same Eulerian description is adopted at the macroscale for the kinematic of all the constituents of the mixture. In the current configuration, at time t, the velocity of the skeleton particle will be noted $\mathbf{v}^s(\mathbf{x}, t)$ and that of the fluid particle $\mathbf{v}^f(\mathbf{x}, t)$. They are related to the velocities $\mathbf{v}_{(\alpha)}(\mathbf{z}, t)$ by

$$\rho_a^{\alpha}\mathbf{v}^{\alpha}(\mathbf{x}, t) = \langle \rho_{(\alpha)}\mathbf{v}_{(\alpha)}(\mathbf{z}, t) \rangle^{\alpha} \quad \alpha = s \text{ or } f \tag{5}$$

The particulate derivative $\mathrm{d}_{\alpha}(\cdot)/\mathrm{d}t$ corresponds to a total time derivative taken by following the considered particle $\alpha = s$ or f in its motion

$$\frac{\mathrm{d}_{\alpha}(\cdot)}{\mathrm{d}t} = \frac{\partial(\cdot)}{\partial t} + \mathrm{grad}_{\mathbf{x}}(\cdot) \cdot \mathbf{v}^{\alpha} \quad \alpha = s \text{ or } f \tag{6}$$

where the operator $\mathrm{grad}_{\mathbf{x}}$ and $\partial/\partial t$ are defined by (3) and (4). As in classical continuum mechanics, we can introduce a deformation rate tensor $\mathbf{d}^{\alpha}$, referred to the current configuration and defined for each medium $\alpha = s$ or f considered separately

$$\mathbf{d}^{\alpha} = \frac{1}{2}(\mathrm{grad}_{\mathbf{x}}\, \mathbf{v}^{\alpha} + {}^{t}\mathrm{grad}_{\mathbf{x}}\, \mathbf{v}^{\alpha}), \quad d_{ij}^{\alpha} = \frac{1}{2}\left(\frac{\partial v_i^{\alpha}}{\partial x_j} + \frac{\partial v_j^{\alpha}}{\partial x_i}\right), \quad \mathrm{tr}\,\mathbf{d}^{\alpha} = d_{ii}^{\alpha} = \mathrm{div}\,\mathbf{v}^{\alpha} \tag{7}$$

Furthermore, let us recall that $\mathrm{div}\,\mathbf{v}^{\alpha}$ represents the rate of volume dilatation calculated by following the volume $\mathrm{d}\Omega_t$ in the motion and deformation of the continuum $\alpha = s$ or f

$$\frac{1}{d\Omega_t}\frac{d_\alpha}{dt}(d\Omega_t) = \operatorname{tr}\mathbf{d}^\alpha = d^\alpha_{ii} = \operatorname{div}\mathbf{v}^\alpha \tag{8}$$

2.4. *Mass conservation*

The mass balance equation for the fluid and the solid at the microscale is

$$\frac{\partial \rho_{(\alpha)}}{\partial t} + \operatorname{div}_{\mathbf{z}}(\rho_{(\alpha)}\mathbf{v}_{(\alpha)}) = 0 \quad \alpha = s \text{ or } f \tag{9}$$

Equation (4) with $\psi_{(\alpha)} = \rho_{(\alpha)}$ and (3) with $\psi_{(\alpha)} = \rho_{(\alpha)}\mathbf{v}_{(\alpha)}$ (the validity of which requires that there is no creation nor transfer of mass) yield

$$\frac{\partial}{\partial t}\rho^\alpha_a = \left\langle \frac{\partial}{\partial t}\rho_{(\alpha)} \right\rangle^\alpha + \int_{\Gamma^\alpha} f(\mathbf{x}-\mathbf{z})\rho_{(\alpha)}(\mathbf{z},t)\mathbf{v}_{(\alpha)}(\mathbf{z},t)\cdot\mathbf{n}_{(\alpha)}(\mathbf{z},t)\,dS \tag{10}$$

$$\operatorname{div}_{\mathbf{x}}(\rho^\alpha_a\mathbf{v}^\alpha) = \langle \operatorname{div}_{\mathbf{z}}(\rho_{(\alpha)}\mathbf{v}_{(\alpha)})\rangle^\alpha - \int_{\Gamma^\alpha} f(\mathbf{x}-\mathbf{z})\rho_{(\alpha)}(\mathbf{z},t)\mathbf{v}_{(\alpha)}(\mathbf{z},t)\cdot\mathbf{n}_{(\alpha)}(\mathbf{z},t)\,dS \tag{11}$$

Adding (10) to (11) yields the mass balance equation at the macroscale for the fluid and the skeleton

$$\frac{\partial \rho^f_a}{\partial t} + \operatorname{div}_{\mathbf{x}}(\rho^f_a\mathbf{v}^f) = 0 \tag{12}$$

$$\frac{\partial \rho^s_a}{\partial t} + \operatorname{div}_{\mathbf{x}}(\rho^s_a\mathbf{v}^s) = 0 \tag{13}$$

Using (6), (12) and (13) can be written in the equivalent form

$$\frac{d_f\rho^f_a}{dt} + \rho^f_a \operatorname{div}_{\mathbf{x}}\mathbf{v}^f = 0 \tag{14}$$

$$\frac{d_s\rho^s_a}{dt} + \rho^s_a \operatorname{div}_{\mathbf{x}}\mathbf{v}^s = 0 \tag{15}$$

Equations (8), (14) and (15) yield

$$\frac{d_f}{dt}(\rho^f_a\,d\Omega_t) = 0 \tag{16}$$

$$\frac{d_s}{dt}(\rho^s_a\,d\Omega_t) = 0 \tag{17}$$

2.5. *Balance of linear momentum*

The local form of the balance of linear momentum for the fluid and the solid at the microscale is

$$\operatorname{div}_{\mathbf{z}}\boldsymbol{\sigma}_{(\alpha)} + \rho_{(\alpha)}(\mathbf{b}_{(\alpha)} - \gamma_{(\alpha)}) = 0 \quad \alpha = s \text{ or } f \tag{18}$$

where $\boldsymbol{\sigma}_{(\alpha)}(\mathbf{z},t)$ and $\gamma_{(\alpha)}(\mathbf{z},t)$, respectively, denote the Cauchy stress tensor and the acceleration relative to the constituent α. Also $\mathbf{b}_{(\alpha)}$ corresponds to an external body force which, in the following, will be assumed to be uniform and identical for the two constituents and will thus be denoted $\mathbf{b}$.

Using (3) with $\psi_{(\alpha)} = \boldsymbol{\sigma}_{(\alpha)}$, it is readily seen that

$$\mathrm{div}_{\mathbf{x}} \langle \boldsymbol{\sigma}_{(\alpha)} \rangle^{\alpha} = \langle \mathrm{div}_{\mathbf{z}}\, \boldsymbol{\sigma}_{(\alpha)} \rangle^{\alpha} - \varepsilon^{\alpha} \mathbf{f}^{\mathrm{int}} \quad \varepsilon^{f} = +1, \quad \varepsilon^{s} = -1 \tag{19}$$

where

$$\mathbf{f}^{\mathrm{int}}(\mathbf{x}, t) = \int_{\Gamma^{f}} f(\mathbf{x}-\mathbf{z}) \boldsymbol{\sigma}_{(f)}(\mathbf{z}, t) \cdot \mathbf{n}_{(f)}(\mathbf{z}, t)\, \mathrm{d}S \tag{20}$$

Equation (20) reveals that the vector $\mathbf{f}^{\mathrm{int}}\, \mathrm{d}\Omega_t$ can be interpreted at the micro-level as a drag interaction exerted within the infinitesimal volume $\mathrm{d}\Omega_t$, at the interface between the solid matrix and the fluid. Combining (18), where the acceleration term is first neglected, with (19) yields

$$\mathrm{div}_{\mathbf{x}} \langle \boldsymbol{\sigma}_{(\alpha)} \rangle^{\alpha} + \rho_a^{\alpha} \mathbf{b} + \varepsilon^{\alpha} \mathbf{f}^{\mathrm{int}} = 0 \quad \varepsilon^{\alpha} = \pm 1$$

which allows to interpret $\langle \boldsymbol{\sigma}_{(\alpha)} \rangle^{\alpha}$ as the partial Cauchy stress tensor $\boldsymbol{\sigma}^{\alpha}$ relative to the constituent α. The vector $\mathbf{f}^{\mathrm{int}}$ now appears at the macro-level as a skeleton–fluid interaction body force while a force of opposite sign is exerted by the fluid upon the skeleton.

We now assume that the constitutive law of the fluid at the microscale is given by

$$\boldsymbol{\sigma}_{(f)}(\mathbf{z}, t) = -\pi(\mathbf{z}, t)\mathbf{1} + \boldsymbol{\tau}(\mathbf{z}, t) \quad \boldsymbol{\tau} = \lambda\, \mathrm{div}_{\mathbf{z}}\, \mathbf{v}_{(f)} \mathbf{1} + \mu(\mathrm{grad}_{\mathbf{z}}\, \mathbf{v}_{(f)} + {}^{t}\mathrm{grad}_{\mathbf{z}}\, \mathbf{v}_{(f)}) \tag{21}$$

where $\pi(\mathbf{z}, t)$ denotes the fluid pressure at the microscale, λ and μ the viscosity coefficients. In some cases, the mean viscous stress $\langle \boldsymbol{\tau} \rangle^{\alpha}$ can be explicitly determined, which makes it possible to evaluate this term quantitatively. For instance, when the fluid is incompressible, and the solid is a granular medium made of rigid grains, it can be shown that $\langle \boldsymbol{\tau} \rangle^{\alpha} = 2\mu \mathbf{d}(\mathbf{u})$, where $\mathbf{d}(\mathbf{u})$ is the deformation rate tensor associated to the fictitious mean velocity $\mathbf{u} = \phi^{s} \mathbf{u}^{s} + \phi^{f} \mathbf{u}^{f}$. In the following, we neglect, however, the mean viscous stress $\langle \boldsymbol{\tau} \rangle^{\alpha}$ so that the partial stress $\boldsymbol{\sigma}^{f}$ takes the form

$$\boldsymbol{\sigma}^{f} = -\phi p \mathbf{1} \tag{22}$$

where $p = \langle \pi \rangle^{\alpha} / \phi$ corresponds to the mean pressure over the volume actually occupied by the fluid at the microscale.

When inertia effects cannot be neglected, the usual acceleration term $\rho_a^{\alpha} \gamma^{\alpha}$ appears in the macroscale form of the balance momentum equation for each constituent

$$\mathrm{div}\, \boldsymbol{\sigma}^{f} + \rho_a^{f}(\mathbf{b} - \gamma^{f}) + \mathbf{f}^{\mathrm{int}} = 0 \tag{23}$$

$$\mathrm{div}\, \boldsymbol{\sigma}^{s} + \rho_a^{s}(\mathbf{b} - \gamma^{s}) - \mathbf{f}^{\mathrm{int}} = 0 \tag{24}$$

However, it can be shown that the partial stresses now differ from the mean value of the microscale stress according to

$$\boldsymbol{\sigma}^{\alpha} = \langle \boldsymbol{\sigma}_{(\alpha)} \rangle^{\alpha} - \langle \rho_{(\alpha)} \mathbf{v}'_{(\alpha)} \otimes \mathbf{v}'_{(\alpha)} \rangle \tag{25}$$

where $\mathbf{v}'_{(\alpha)} = \mathbf{v}_{(\alpha)} - \mathbf{v}^{\alpha}$. As a matter of fact, it is recalled that the acceleration γ^{α} is the particulate derivative of the velocity $\mathbf{v}^{\alpha}$ defined by (5), see (6). The correcting term $\langle \rho_{(\alpha)} \mathbf{v}'_{(\alpha)} \otimes \mathbf{v}'_{(\alpha)} \rangle$ in (25) results from the fact that the macroscale dynamic force $\rho_a^{\alpha} \gamma^{\alpha}\, \mathrm{d}\Omega_t$ is not equal to the mean dynamic force over the constituent α, i.e. $\langle \rho_{(\alpha)} \gamma_{(\alpha)} \rangle^{\alpha}\, \mathrm{d}\Omega_t$.

In mixture theories, Darcy's law is generally introduced here, mainly as an empirical law allowing the identification of the interaction force $\mathbf{f}^{\mathrm{int}}$ (e.g., Morland, 1992). In the context of macroscale theories, such an introduction of Darcy's law is premature. As shown later on, Darcy's law in such theories is associated with the overall dissipation and appears as a conduction law in the same way as Fourier's law. However, in view of the balance of energy involved in thermodynamics, it is worthwhile to derive the balance of mechanical energy. Multiply (23) by $\mathbf{v}^f$ and (24) by $\mathbf{v}^s$, integrate by parts over a domain Ω_t, then sum the two resulting equations. Using in addition (7) and the symmetry of the partial stress tensors, one finally obtains the mechanical energy theorem in the form

$$\overset{\circ}{\mathcal{K}} + \overset{\circ}{\mathcal{W}}_d = \overset{\circ}{\mathcal{W}}_{\text{ext}} \tag{26}$$

where

$$\overset{\circ}{\mathcal{K}} = \frac{1}{2}\int_{\Omega_t}\left[\rho_a^f\frac{d_f}{dt}(\mathbf{v}^f\cdot\mathbf{v}^f) + \rho_a^s\frac{d_s}{dt}(\mathbf{v}^s\cdot\mathbf{v}^s)\right]\mathrm{d}\Omega_t \tag{27}$$

$$\overset{\circ}{\mathcal{W}}_d = \int_{\Omega_t}[\mathrm{tr}(\boldsymbol{\sigma}^f\mathbf{d}^f) + \mathrm{tr}(\boldsymbol{\sigma}^s\mathbf{d}^s) + \mathbf{f}^{\text{int}}\cdot(\mathbf{v}^s - \mathbf{v}^f)]\,\mathrm{d}\Omega_t \tag{28}$$

$$\overset{\circ}{\mathcal{W}}_{\text{ext}} = \int_{\Omega_t}(\rho_a^f\mathbf{b}\cdot\mathbf{v}^f + \rho_a^s\mathbf{b}\cdot\mathbf{v}^s)\,\mathrm{d}\Omega_t + \int_{\partial\Omega_t}(\mathbf{T}^f\cdot\mathbf{v}^f + \mathbf{T}^s\cdot\mathbf{v}^s)\,\mathrm{d}a \tag{29}$$

where we noted $\mathbf{T}^f = \boldsymbol{\sigma}^f\cdot\mathbf{n}$ and $\mathbf{T}^s = \boldsymbol{\sigma}^s\cdot\mathbf{n}$, the stress vectors acting at the boundary $\partial\Omega_t$ of Ω_t, as external surface forces on the fluid and on the skeleton, respectively. According to theorem (26), the work increment $\overset{\circ}{\mathcal{W}}_{\text{ext}}\,\mathrm{d}t$ provided during time $\mathrm{d}t$ by the external forces to both constituents considered as a whole is transformed both in kinetic energy $\overset{\circ}{\mathcal{K}}\,\mathrm{d}t$ and in overall stress work $\overset{\circ}{\mathcal{W}}_d\,\mathrm{d}t$. The overall stress power $\overset{\circ}{\mathcal{W}}_d$ can be written as the integral of the overall stress power density $\overset{\circ}{w}_d$ defined by

$$\overset{\circ}{w}_d = \mathrm{tr}(\boldsymbol{\sigma}^f\mathbf{d}^f) + \mathrm{tr}(\boldsymbol{\sigma}^s\mathbf{d}^s) + \mathbf{f}^{\text{int}}\cdot(\mathbf{v}^s - \mathbf{v}^f) \tag{30}$$

Note that $\overset{\circ}{w}_d$ accounts for the power associated not only with deformation of the particles (term $(\mathrm{tr}(\boldsymbol{\sigma}^f\mathbf{d}^f) + \mathrm{tr}(\boldsymbol{\sigma}^s\mathbf{d}^s))$, but also with the interaction force $\mathbf{f}^{\text{int}}$, which acts as an internal force when considering the porous medium as a whole (term $\mathbf{f}^{\text{int}}\cdot(\mathbf{v}^s - \mathbf{v}^f)$). Note also that $\overset{\circ}{\mathcal{K}}$ represents the exact time derivative of the kinetic energy of the whole matter contained in volume Ω_t and time t and followed in the material motion since, according to (16) and (17), whatever g we have

$$\rho_a^f\frac{d_f g_f}{\mathrm{d}t}\mathrm{d}\Omega_t = \frac{d_f}{\mathrm{d}t}(\rho_a^f g_f\,\mathrm{d}\Omega_t) \quad \rho_a^s\frac{d_s g_s}{\mathrm{d}t}\mathrm{d}\Omega_t = \frac{d_s}{\mathrm{d}t}(\rho_a^s g_s\,\mathrm{d}\Omega_t) \tag{31}$$

2.6. *The first and second laws*

The first law expresses that the energy is conserved when considering it in all its forms. Assume here, for the sake of simplicity, that the energy is externally supplied only in form of heat or in form of mechanical work. Then the first law, when applied to the porous medium considered as a whole, formally reads

$$\overset{\circ}{\mathcal{E}} + \overset{\circ}{\mathcal{K}} = \overset{\circ}{\mathcal{W}}_{\text{ext}} - \overset{\circ}{\mathcal{Q}} \tag{32}$$

In the above equation, $\overset{\circ}{\mathcal{E}}$ represents the rate of internal energy of the whole matter contained at time t in the volume Ω_t, while $\overset{\circ}{\mathcal{Q}}$ is the rate of external heat supply. In mixture theories, the above overall form of the first law is generally justified using averaging procedures on each constituent and summation.

Noting e_f and e_s the density of internal energy per mass unit associated, respectively, with the fluid and the skeleton particles, on account in particular of (31) with $g = e$, $\overset{\circ}{\mathcal{E}}$ can be expressed in the form

$$\overset{\circ}{\mathscr{E}} = \int_{\Omega_t} \left(\rho_a^f \frac{d_f e_f}{dt} + \rho_a^s \frac{d_s e_s}{dt} \right) d\Omega_t \tag{33}$$

Assuming, for the sake of simplicity, that the heat is externally supplied only through heat conduction, and denoting by $\mathbf{q}$ the overall heat flux vector, $\overset{\circ}{\mathscr{Q}}$ can be simply expressed as

$$\overset{\circ}{\mathscr{Q}} = -\int_{\partial\Omega_t} \mathbf{q}\cdot\mathbf{n}\, da = -\int_{\Omega_t} \operatorname{div}\mathbf{q}\, d\Omega_t \tag{34}$$

Combining (26) and (32), it follows

$$\overset{\circ}{\mathscr{E}} = \overset{\circ}{\mathscr{W}}_d + \overset{\circ}{\mathscr{Q}} \tag{35}$$

Altogether, (28), (35), (33) and (34) finally yields the local form of the first law

$$\rho_a^f \frac{d_f e_f}{dt} + \rho_a^s \frac{d_s e_s}{dt} = \operatorname{tr}(\boldsymbol{\sigma}^f \cdot \mathbf{d}^f) + \operatorname{tr}(\boldsymbol{\sigma}^s \cdot \mathbf{d}^s) + \mathbf{f}^{\text{int}} \cdot (\mathbf{v}^s - \mathbf{v}^f) - \operatorname{div}\mathbf{q} \tag{36}$$

The second law expresses that the entropy rate of any material system cannot be less than the rate of external supply of entropy. As it is well known, the so-called associated spontaneous creation of entropy is related to the spontaneous deterioration of energy, i.e. to its decreasing ability to be converted into efficient mechanical work. Assuming again that the external supply of entropy is due to heat conduction only (i.e. there is no heat sources) and noting s_f and s_s the density of internal entropy per mass unit, respectively, of the fluid particle and of the skeleton particle, the second law, applied to the porous medium considered as a whole reads

$$\int_{\Omega_t} \left(\rho_a^f \frac{d_f s_f}{dt} + \rho_a^s \frac{d_s s_s}{dt} \right) d\Omega_t \geqslant \int_{\partial\Omega_t} -\frac{\mathbf{q}\cdot\mathbf{n}}{T}\, da \tag{37}$$

where (31) has been used with $g = s$. In the above equation, T is the temperature of all the matter contained in the elementary volume $d\Omega_t$ assumed to be in thermal equilibrium. Applying the divergence theorem and arguing that the temperature is positive, the local form of (37) is

$$\rho_a^f \frac{d_f s_f}{dt} + \rho_a^s \frac{d_s s_s}{dt} = -\operatorname{div}\frac{\mathbf{q}}{T} + \frac{\varphi}{T} \quad \varphi \geqslant 0 \tag{38}$$

Through (38) φ/T is defined as the internal production rate of entropy and has to be positive, while φ is the associated dissipation.

In addition to the boundary conditions, the final set of equations to be solved is the set of field equations consisting of the continuity eqns (12) and (13); the momentum balance eqns (23) and (24) for the fluid and the skeleton, to which Darcy's law is added in order to identify the interaction force as mentioned earlier; the energy eqn (33), to which Fourier's law is added in order to describe the heat conduction phenomenon. The second law is generally assumed to be implicitly satisfied, when finally specifying the explicit expressions of the internal energies and the constitutive relations linking the partial stresses to the strain and to the temperature. Their choice is the last step in elaborating the theory and differs from one author to another.

3. FROM MIXTURE THEORIES TO A MACROSCOPIC LAGRANGIAN APPROACH

The mixture theory approach has been outlined in the specific case of a one fluid–solid mixture. The macroscale approach, when predicting the strain of the skeleton, differs from the mixture theory in regards to two points. First, the roles played by the different constituents are not symmetric anymore. Since particular attention is paid to the skeleton, all the physical balance laws are referred to the kinematics of the skeleton, while favoring their formulation in a Lagrangian approach with respect to the latter (Biot, 1977; Coussy, 1989). Second, the material elementary volumes $d\Omega_t$ (the set of which constitutes the porous medium) are considered at each time as open thermodynamic systems. Indeed, when followed in the movement and the deformation of the skeleton, a material elementary volume $d\Omega_t$ exchanges fluid mass with the outside. Therefore, at the macro-level at which these open systems are defined, no distinction is to be made between the fluid and the skeleton constituents when formulating their constitutive equations. In fact this distinction is relevant only at the microscale which constitutes the starting level of the mixture theories. In what follows, starting from the equations derived from mixture theory, we show the existence of a thermodynamic potential for the open system upon which the formulation of the constitutive equations can be firmly grounded.

3.1. *Deformation and kinematics, Lagrangian approach*

In contrast to the Eulerian approach which refers only to the current description, the Lagrangian description introduces a reference initial configuration, associated here with the skeleton. In fact, the skeleton particle located at point $\mathbf{x}$ with volume $d\Omega_t$ at time t was located at point $\mathbf{X}$ with volume $d\Omega_0$ at time $t = 0$. We write

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \tag{39}$$

From now on, the divergence and the gradient with respect to $\mathbf{x}$ (resp. $\mathbf{X}$) will be denoted div and grad (resp. Div and Grad). Let $\mathbf{F}$ be the skeleton deformation gradient, mapping the material skeleton line element $d\mathbf{X}$ onto the current line $d\mathbf{x}$ through

$$d\mathbf{x} = \mathbf{F}\cdot d\mathbf{X} \quad \text{where } \mathbf{F} = \operatorname{Grad}\mathbf{x} = \frac{\partial x_i}{\partial X_j}\mathbf{e}_i \otimes \mathbf{e}_j \tag{40}$$

The current volume $d\Omega_t$ and the initial volume $d\Omega_0$ are linked by

$$d\Omega_t = \det\mathbf{F}\, d\Omega_0 = J\, d\Omega_0 \tag{41}$$

Introducing the skeleton displacement vector $\xi = \mathbf{x} - \mathbf{X}$, we have

$$\mathbf{F} = 1 + \operatorname{Grad}\xi \tag{42}$$

In the Lagrangian approach, the deformation is described with the help of the symmetric Green–Lagrange strain tensor Δ defined by

$$d\mathbf{x}\cdot d\mathbf{x}' - d\mathbf{X}\cdot d\mathbf{X}' = 2\, d\mathbf{X}\cdot\Delta\cdot d\mathbf{X}' \Leftrightarrow 2\Delta = {}^t\mathbf{F}\cdot\mathbf{F} - 1 \tag{43}$$

Using (42), an equivalent form of Δ in terms of the displacement vector is

$$2\Delta = \operatorname{Grad}\xi + {}^t\operatorname{Grad}\xi + {}^t\operatorname{Grad}\xi\cdot\operatorname{Grad}\xi \tag{44}$$

Furthermore, using (7), (40) and (43) and the definition $\mathbf{v}^s = (d_s\mathbf{x}/dt)$, it follows that the Green–Lagrange strain tensor rate is linked to the Eulerian strain rate tensor through

$$\mathbf{d}^s = {}^t\mathbf{F}^{-1}\cdot\frac{d_s\Delta}{dt}\cdot\mathbf{F}^{-1} \tag{45}$$

The skeleton infinitesimal material surface da, having $\mathbf{n}$ as unit normal and centred at point $\mathbf{x}$ at time t, was the surface dA, having $\mathbf{N}$ as unit normal and centred at point $\mathbf{X}$ at time

$t = 0$. Hence, due to (40), the initial volume $\mathrm{d}\Omega_0 = \mathbf{N}\cdot \mathrm{d}\mathbf{X}\,\mathrm{d}A$ becomes after deformation $\mathrm{d}\Omega_t = \mathbf{n}\cdot\mathbf{F}\cdot\mathrm{d}\mathbf{X}\,\mathrm{d}a$. From (41) it follows that

$$ {}^t\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}a = J\mathbf{N}\,\mathrm{d}A \qquad (46)$$

To any Eulerian velocity vector $\mathbf{v}(\mathbf{x}, t)$, we can therefore associate the Lagrangian velocity vector $\mathbf{V}(\mathbf{X}, t)$ such that

$$\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}a = \mathbf{V}\cdot\mathbf{N}\,\mathrm{d}a \qquad (47)$$

From (47), (46) and the divergence theorem, it follows that

$$\mathbf{v}\,\mathrm{d}\Omega_t = \mathbf{V}\cdot{}^t\mathbf{F}\,\mathrm{d}\Omega_0 \quad \mathrm{div}\,\mathbf{v}\,\mathrm{d}\Omega_t = \mathrm{Div}\,\mathbf{V}\,\mathrm{d}\Omega_0 \quad \text{where } \mathrm{Div}\,\mathbf{V} = \frac{\partial V_i}{\partial X_i} \qquad (48)$$

In particular let $\mathbf{w}(\mathbf{x}, t)$ be the relative fluid mass flow rate vector with respect to the skeleton movement

$$\mathbf{w} = \rho^f_a(\mathbf{v}^f - \mathbf{v}^s) \qquad (49)$$

According to (47) and (48), we can define the Lagrangian relative of fluid mass flow rate vector $\mathbf{M}(\mathbf{X}, t)$ (Biot, 1977) as

$$\mathbf{w}\cdot\mathbf{n}\,\mathrm{d}a = \mathbf{M}\cdot\mathbf{N}\,\mathrm{d}A \quad \mathbf{w}\,\mathrm{d}\Omega_t = \mathbf{M}\cdot{}^t\mathbf{F}\,\mathrm{d}\Omega_0 \quad \mathrm{div}\,\mathbf{w}\,\mathrm{d}\Omega_t = \mathrm{Div}\,\mathbf{M}\,\mathrm{d}\Omega_0 \qquad (50)$$

To any Eulerian volume density $g(\mathbf{x}, t)$, we can associate the Lagrangian volume density $G(\mathbf{X}, t)$

$$g(\mathbf{x}, t)\,\mathrm{d}\Omega_t = G(\mathbf{X}, t)\,\mathrm{d}\Omega_0 \Leftrightarrow Jg = G \qquad (51)$$

From (6), (8), and (51), it follows that

$$\left(\frac{d_s g}{\mathrm{d}t} + g\,\mathrm{div}\,\mathbf{v}^s\right)\mathrm{d}\Omega_t = \frac{d_s G}{\mathrm{d}t}\mathrm{d}\Omega_0 \qquad (52)$$

where $d_s G/\mathrm{d}t$ corresponds to the partial derivative of G with respect to t. Consider now the expression $\rho^f_a(d_f g_f/\mathrm{d}t) + \rho^s_a(d_s g_s/\mathrm{d}t)$. We can write

$$\rho^f_a\frac{d_f g_f}{\mathrm{d}t} + \rho^s_a\frac{d_s g_s}{\mathrm{d}t} = \frac{d_f}{\mathrm{d}t}(\rho^f_a g_f) - g_f\frac{d_f}{\mathrm{d}t}(\rho^f_a) + \frac{d_s}{\mathrm{d}t}(\rho^s_a g_s) - g_s\frac{d_s}{\mathrm{d}t}(\rho^s_a) \qquad (53)$$

Using the definition (6) of the particulate derivative, definition (49) of the fluid mass flow rate vector $\mathbf{w}$, together with the continuity eqns (14) and (15), (53) can be rewritten in the equivalent form

$$\rho^f_a\frac{d_f g_f}{\mathrm{d}t} + \rho^s_a\frac{d_s g_s}{\mathrm{d}t} = \frac{d_s g}{\mathrm{d}t} + g\,\mathrm{div}\,\mathbf{v}^s + \mathrm{div}(\mathbf{w}g_f) \qquad (54)$$

where g is defined as a density associated with the material porous elementary volume $\mathrm{d}\Omega_t$ viewed as a whole, such that

$$g\,\mathrm{d}\Omega_t = (\rho^f_a g_f + \rho^s_a g_s)\,\mathrm{d}\Omega_t \qquad (55)$$

Combining (50), (51), (52), (54) and (55), we obtain

$$\left(\rho_a^f \frac{d_f g_f}{\mathrm{d}t} + \rho_a^s \frac{d_s g_s}{\mathrm{d}t}\right) \mathrm{d}\Omega_t = \left[\frac{d_s G}{\mathrm{d}t} + \mathrm{Div}(\mathbf{M} g_f)\right] \mathrm{d}\Omega_0 \quad \text{where } g\,\mathrm{d}\Omega_t = G\,\mathrm{d}\Omega_0 \tag{56}$$

This relation transforms the Eulerian quantity of the left hand member into the Lagrangian quantity of the right member. It will prove to be very useful in what follows.

3.2. *The physical balance laws, Lagrangian approach*

3.2.1. *The Lagrangian fluid mass content variation.* Expressing the fluid mass conservation eqn (14) with reference to the skeleton movement yields

$$\frac{d_s \rho_a^f}{\mathrm{d}t} + \rho_a^f \,\mathrm{div}\, \mathbf{v}^s + \mathrm{div}\, \mathbf{w} = 0 \tag{57}$$

The left hand member of (57) is equal to the right hand member of (54) with $g_f = 1, g_s = 0$ and $g = \rho^f \phi$. Therefore, applying the general result with the previous identification, we obtain

$$\frac{d_s m}{\mathrm{d}t} + \mathrm{Div}\, \mathbf{M} = 0 \quad m = J\rho_a^f - \rho_{a,0}^f \tag{58}$$

where the subscript 0 refers to initial values. In view of its definition, m is a Lagrangian density such that $m(\mathbf{X}, t)\,\mathrm{d}\Omega_0$ represents the fluid mass variation (>0 if added, <0 if extracted) which is experienced by the elementary volume $\mathrm{d}\Omega_t$ from time $t = 0$. It was first introduced by Biot (1972).

The mass conservation of the porous medium viewed as a whole and referred to the skeleton movement is obtained by summing (15), (57)

$$\frac{d_s r}{\mathrm{d}t} + r\,\mathrm{div}\, \mathbf{v}^s + \mathrm{div}\, \mathbf{w} = 0 \tag{59}$$

where r is the total mass density

$$r = \rho_a^s + \rho_a^f \tag{60}$$

The left hand member of (59) is equal to the right hand member of (54) with $g_f = g = 1$ and $g = r$. Therefore, applying the general result with the previous identification, we obtain

$$\frac{d_s R}{\mathrm{d}t} + \mathrm{Div}\, \mathbf{M} = 0 \quad R = Jr \tag{61}$$

In view of its definition, R is a Lagrangian density such that $R(\mathbf{X}, t)\,\mathrm{d}\Omega_0$ represents the fluid mass currently contained in the elementary volume $\mathrm{d}\Omega_t$. Combining (58) and (61) and integrating gives

$$R = r_0 + m \tag{62}$$

which expresses that the variation of mass of the elementary volume $\mathrm{d}\Omega_t$ followed from time $t = 0$ in the skeleton movement is only due to the fluid mass it has exchanged with the outside. Note the simplicity of the Lagrangian continuity eqns (58) and (62) when compared to the Eulerian ones (57) and (59).

3.2.2. *The Lagrangian balance of momentum.* The balance of momentum of the porous medium viewed as a whole is obtained by summing (23) and (24)

$$\operatorname{div} \boldsymbol{\sigma} + r\mathbf{b} - \rho_a^f \gamma^f - \rho_a^s \gamma^s = 0 \tag{63}$$

where $\boldsymbol{\sigma}$ is the total stress

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^f + \boldsymbol{\sigma}^s \tag{64}$$

The overall stress power $\mathring{w}_d \mathrm{d}\Omega_t$, given by (30), can be written in a different Lagrangian form. Using (64) and (22), (30) yields

$$\mathring{w}_d \, \mathrm{d}\Omega_t = [\operatorname{tr}(\boldsymbol{\sigma} \cdot \mathbf{d}^s) - \phi p \operatorname{div}(\mathbf{v}^f - \mathbf{v}^s) + \mathbf{f}^{\text{int}} \cdot (\mathbf{v}^s - \mathbf{v}^f)] \, \mathrm{d}\Omega_t \tag{65}$$

Using (22), (23) and (49), it follows that

$$\mathring{w}_d \, \mathrm{d}\Omega_t = \left[\operatorname{tr}(\boldsymbol{\sigma} \cdot \mathbf{d}^s) - \operatorname{div}\left(\frac{p}{\rho^f} \mathbf{w}\right) + \mathbf{w} \cdot \left(\mathbf{b} - \frac{d_f \mathbf{v}^f}{\mathrm{d}t}\right) \right] \mathrm{d}\Omega_t \tag{66}$$

The local balance of momentum can be expressed in a Lagrangian form, after introducing the Piola–Kirchhoff stress tensor $\boldsymbol{\pi}$. The latter is defined by referring the stress power associated with the total stress tensor to the initial volume

$$\operatorname{tr}(\boldsymbol{\sigma} \cdot \mathbf{d}^s) \, \mathrm{d}\Omega_t = \operatorname{tr}\left(\boldsymbol{\pi} \cdot \frac{d_s \Delta}{\mathrm{d}t}\right) \mathrm{d}\Omega_0 \tag{67}$$

where $d_s\Delta/\mathrm{d}t$ is the rate of the Green–Lagrange strain tensor, see (45). Using the definition (7) of $\mathbf{d}^s$, (45) linking $\mathbf{d}^s$ and $d_s\Delta/\mathrm{d}t$ and (67) defining $\boldsymbol{\pi}$, the relation relying $\boldsymbol{\sigma}$ and $\boldsymbol{\pi}$ is

$$\boldsymbol{\pi} = J(\mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \cdot {}^t\mathbf{F}^{-1}) \tag{68}$$

After some calculations (Coussy, 1989, 1995), the Lagrangian expression of the local form of momentum balance is found to be

$$\operatorname{Div}(\mathbf{F} \cdot \boldsymbol{\pi}) + (r_0 + m)\mathbf{b} - (\rho_{a,0}^f + m)\gamma^f - \rho_{a,0}^s \gamma^s = 0 \tag{69}$$

The Lagrangian form of the stress power density can also be derived from (66), (67) and (50)

$$\begin{aligned} &\left[\operatorname{tr}(\boldsymbol{\sigma} \cdot \mathbf{d}^s) - \operatorname{div}\left(\frac{p}{\rho^f} \mathbf{w}\right) + \mathbf{w} \cdot (\mathbf{b} - \gamma^f) \right] \mathrm{d}\Omega_t \\ &\qquad = \left[\operatorname{tr}\left(\boldsymbol{\pi} \cdot \frac{d_s \Delta}{\mathrm{d}t}\right) - \operatorname{Div}\left(\frac{p}{\rho^f} \mathbf{M}\right) + \mathbf{M}^t \cdot \mathbf{F} \cdot (\mathbf{b} - \gamma^f) \right] \mathrm{d}\Omega_0 \end{aligned} \tag{70}$$

3.2.3. *The first and second laws, Lagrangian approach.* Multiplying (33) by $\mathrm{d}\Omega_t$ and transforming the left hand member of the resulting equation by using (56) with $g = e$, and its right hand member by using (70), we obtain the Lagrangian local form of the energy balance equation

$$\frac{d_s E}{\mathrm{d}t} = \operatorname{tr}\left(\boldsymbol{\pi} \cdot \frac{d_s \Delta}{\mathrm{d}t}\right) - \operatorname{Div}\left[\left(e_f + \frac{p}{p^f}\right)\mathbf{M}\right] - \operatorname{Div} \mathbf{Q} + \mathbf{M} \cdot {}^t\mathbf{F} \cdot (\mathbf{b} - \gamma^f) \tag{71}$$

Similarly, from the Eulerian eqn (38) the Lagrangian local form of the second law is

$$\frac{d_s S}{\mathrm{d}t} = -\mathrm{Div}(\mathbf{M}s_f) - \mathrm{Div}\frac{\mathbf{Q}}{T} + \frac{\Phi}{T} \quad \Phi \geqslant 0 \tag{72}$$

In the two above equations, conformly to (47) and (56), we let

$$E\,\mathrm{d}\Omega_0 = [\rho^f_a e_f + \rho^s_a e_s]\,\mathrm{d}\Omega_t \quad S\,\mathrm{d}\Omega_0 = [\rho^f_a s_f + \rho^s_a s_s]\,\mathrm{d}\Omega_t \tag{73}$$

as also

$$\Phi\,\mathrm{d}\Omega_0 = \varphi\,\mathrm{d}\Omega_t \quad \mathbf{Q}\cdot\mathbf{N}\,\mathrm{d}A = \mathbf{q}\cdot\mathbf{n}\,\mathrm{d}a \tag{74}$$

Note that the Lagrangian densities of internal energy E and of entropy S are volume densities. Since the volume $\mathrm{d}\Omega_t$ is always followed in the movement of the skeleton, except for zero fluid mass exchanges, E and S are never associated with the same matter. Due to their definitions, they are in fact associated with $\mathrm{d}\Omega_t$ viewed as an open system.

The Lagrangian set of eqns (58), (62), (69), (71) and (72) is completely equivalent to the Eulerian set of eqns (12), (13), (63), (33) and (38) with regards to physical laws. However, as shown in the next section, the existence of a macro-potential can be proved only from the Lagrangian formulation.

4. EXISTENCE OF A MACRO Ψ-POTENTIAL

4.1. *The Clausius–Duhem inequality*

Let g_f be the free enthalpy or Gibbs potential per mass unit of the fluid defined as

$$g_f = e_f + \frac{p}{\rho^f} - Ts_f \tag{75}$$

Using the above definition we have the identity

$$\mathrm{Div}\left[\left(\frac{p}{\rho^f} + e_f\right)\mathbf{M}\right] = g_f\,\mathrm{Div}\,\mathbf{M} + \mathbf{M}\cdot{}^t\mathbf{F}\cdot(\mathrm{grad}\,g_f + s_f\,\mathrm{grad}\,T) + T\,\mathrm{Div}(\mathbf{M}s_f) \tag{76}$$

where

$$\mathrm{Grad}(\cdot) = \mathrm{grad}(\cdot)\cdot\mathbf{F} \tag{77}$$

A substitution of (76) into (71), while using (58), yields

$$\mathrm{tr}\left(\boldsymbol{\pi}\cdot\frac{d_s\Delta}{\mathrm{d}t}\right) + g_f\frac{d_s m}{\mathrm{d}t} - \frac{d_s\Psi}{\mathrm{d}t} - \frac{d_s(TS)}{\mathrm{d}t}$$

$$-\mathrm{Div}\,\mathbf{Q} - T\,\mathrm{Div}(\mathbf{M}s_f) + \mathbf{M}\cdot{}^t\mathbf{F}\cdot(-\mathrm{grad}\,g_f - s_f\,\mathrm{grad}\,T + \mathbf{b} - \gamma^f) = 0 \tag{78}$$

where we introduced the Lagrangian free enthalpy per volume unit of the open system $\mathrm{d}\Omega_0$ defined by

$$\Psi = E - TS \tag{79}$$

Furthermore, since $T > 0$, the entropy balance (38) can be written in the alternate form

$$T\frac{d_sS}{\mathrm{d}t} = -T\,\mathrm{Div}(\mathbf{M}s_f) - \mathrm{Div}\,\mathbf{Q} + \frac{\mathbf{Q}}{T}\cdot\mathrm{Grad}\,T + \Phi \quad \Phi \geqslant 0 \tag{80}$$

A combination of (78) and (80) yields the identification of the dissipation Φ in the form

$$\Phi = \Phi_{fl} + \Phi_{th} + \Phi_{l} \geqslant 0 \tag{81}$$

with

$$\Phi_{fl} = \mathbf{M}\cdot{}^t\mathbf{F}\cdot(-\mathrm{grad}\,g_f - s_f\,\mathrm{grad}\,T + \mathbf{b} - \gamma^f) \tag{82}$$

$$\Phi_{th} = -\frac{\mathbf{Q}}{T}\cdot\mathrm{Grad}\,T \tag{83}$$

$$\Phi_l = \mathrm{tr}\left(\pi\frac{d_s\Delta}{\mathrm{d}t}\right) + g_f\frac{d_sm}{\mathrm{d}t} - S\frac{d_sT}{\mathrm{d}t} - \frac{d_s\Psi}{\mathrm{d}t} \tag{84}$$

Note once more that the set of both eqns (80) and (81) are completely equivalent to the set (33) and (38).

4.2. *Identification of dissipations and conduction laws*

The total dissipation Φ appears as the sum of three dissipations Φ_{fl}, Φ_{th} and Φ_l. The dissipation Φ_{fl} appears as the dissipation associated with fluid mass transport, while Φ_{th} is the thermal dissipation associated with heat conduction. Contrary to Φ_{fl} and Φ_{th}, the dissipation Φ_l involves no gradient and solely time derivatives in following the initial elementary system $\mathrm{d}\Omega_0$ in the movement of the skeleton. Independent of the other element adjacent systems, it is called the intrinsic dissipation. Owing to their very different physical nature, although not strictly required by the fundamental inequality (81), the three dissipations are assumed to be positive, independently of each other

$$\Phi_{fl} \geqslant 0 \quad \Phi_{th} \geqslant 0 \quad \Phi_l \geqslant 0$$

In order to express the dissipation associated with fluid mass transport in a more familiar way, let us use the well-known micro state relations of the fluid in terms of Gibbs potential g_f

$$g_f = g_f(p, T) \quad \frac{\partial g_f}{\partial p} = \frac{1}{\rho^f} \quad \frac{\partial g_f}{\partial T} = -s_f \tag{85}$$

A substitution of (85) in (82) yields

$$\Phi_{fl} = \frac{\mathbf{M}}{\rho^f}\cdot{}^t\mathbf{F}\cdot[-\mathrm{grad}\,p + \rho^f(\mathbf{b} - \gamma^f)] \tag{86}$$

Owing to (50) and (77) the corresponding Eulerian dissipations φ_{fl} and φ_{th}, such as $\varphi_{fl\,\mathrm{or}\,th}\,\mathrm{d}\Omega_t = \Phi_{fl\,\mathrm{or}\,th}\,\mathrm{d}\Omega_0$, have also to be positive; hence

$$\varphi_{fl} = \frac{\mathbf{w}}{\rho^f}\cdot(-\mathrm{grad}\,p + \rho^f(\mathbf{b} - \gamma^f)) \geqslant 0 \tag{87}$$

$$\varphi_{th} = -\frac{\mathbf{q}}{T}\cdot\mathrm{grad}\,T \geqslant 0 \tag{88}$$

The dissipations φ_{fl} or φ_{th} thus appear as the product of a force, $-\mathrm{grad}\,p+\rho^f(\mathbf{b}-\gamma^f)$ or $-\mathrm{grad}\,T$, multiplied by a flux, $\mathbf{w}/\rho^f$ or $\mathbf{q}/T$. The simplest conduction laws consist in linearly relating these forces to these fluxes, thus yielding Darcy's law

$$\frac{\mathbf{w}}{\rho^f} = \mathbf{k}\cdot[-\mathrm{grad}\,p+\rho^f(\mathbf{b}-\gamma^f)] \tag{89}$$

and Fourier's law

$$\mathbf{q} = -\boldsymbol{\kappa}\cdot\mathrm{grad}\,T \tag{90}$$

A comparison of (89) and (23), when using (22) and (49) allows *a posteriori* identification of the interaction force $\mathbf{f}^{\mathrm{int}}$ as

$$\mathbf{f}^{\mathrm{int}} = -\phi^2\mathbf{k}^{-1}\cdot(\mathbf{v}^f-\mathbf{v}^s)+p\,\mathrm{grad}\,\phi \tag{91}$$

Hence, Darcy's law, which in the mixture theories is the law required for the above identification of the unknown interaction force $\mathbf{f}^{\mathrm{int}}$, appears now at the same macro level as Fourier's law, as the conduction law of the fluid mass through the open system $\mathrm{d}\Omega_t$. Note that inequalities require for the permeability tensor $\mathbf{k}$ and the thermal conductivity $\boldsymbol{\kappa}$ to be associated with positive quadratic forms.

4.3. *Existence of a macro Ψ-potential, thermoporoelastic behaviour*

The non-negativity of the intrinsic dissipation reads

$$\Phi_I = \pi_{ij}\frac{d_s\Delta_{ij}}{\mathrm{d}t}+g_f\frac{d_s m}{\mathrm{d}t}-S\frac{d_s T}{\mathrm{d}t}-\frac{d_s\Psi}{\mathrm{d}t}\geqslant 0 \tag{92}$$

The dissipation Φ_{fl} is associated with the transport of the fluid mass, i.e., with the viscous behaviour of the fluid. Therefore, Φ_I is the dissipation associated with the solid part of the skeleton, that is the matrix, but expressed in terms of macro variables. Using (81), the thermal eqn (80) reads

$$T\frac{d_s S}{\mathrm{d}t} = -T\,\mathrm{Div}(\mathbf{M}s_f)-\mathrm{Div}\,\mathbf{Q}+\Phi_M \quad \Phi_M = \Phi_{fl}+\Phi_I \tag{93}$$

As defined above, Φ_M represents the mechanical power which is converted into heat rate. It is the total mechanical dissipation and appears as a heat source in thermal eqn (93).

For a thermoelastic matrix, the intrinsic dissipation Φ_I is zero

$$\pi_{ij}\frac{d_s\Delta_{ij}}{\mathrm{d}t}+g_f\frac{d_s m}{\mathrm{d}t}-S\frac{d_s T}{\mathrm{d}t}-\frac{d_s\Psi}{\mathrm{d}t}=0 \tag{94}$$

Obviously, the set of variables (Δ_{ij}, m, T) constitutes a normal set of variables. In other words, each variable of the set can vary independently of the other variables. Since the above equality has to be satisfied always, necessarily

$$\Psi = \Psi(\Delta_{ij}, m, T) \quad \pi_{ij} = \frac{\partial\Psi}{\partial\Delta_{ij}} \quad g_f = \frac{\partial\Psi}{\partial m} \quad S = -\frac{\partial\Psi}{\partial T} \tag{95}$$

The Gibbs potential g_f, in (95), is to be interpreted at the macro-level only as the conjugate variable of m. It is defined as the state variable such that $g_f\,\mathrm{d}m$ represents the energy supplied not in the form of heat between times t and $t+\mathrm{d}t$ to the open macro system $\mathrm{d}\Omega_t$ when increasing its fluid mass content by $\mathrm{d}m$.

Equation (95) allows us to consider the free energy as a macro Ψ-potential, since it is now a function of the set of macro state variables (Δ_{ij}, m, T), while the associate set (π_{ij}, g_f, S) can be derived by taking ordinary derivatives with respect to it. In fact (95) are the state equations of the open system $\mathrm{d}\Omega_t$ viewed as a whole and followed at each time in the skeleton movement. These state equations are the more general form for the constitutive equations of macro thermoporoelasticity. The above thermodynamic approach can be extended to non-elastic behaviours. However such an extension is beyond the scope of this paper. The interested reader can refer to Coussy (1995).

Finally note that the set of Lagrangian equations consisting of the continuity eqns (58), (62), the Lagrangian balance momentum (69), the thermal eqn (93), the state eqns (95), Darcy's and Fourier's laws, (89) and (90), when expressed in terms of Lagrangian vectors **M** and **Q**, constitute a closed set of equations, to be solved when adding the boundary conditions. It constitutes the most general macroscopic continuum approach of thermoporoelasticity before any linearization.

4.4. *Linear thermoporoelasticity, Biot's theory recovered*

In order to recover a more familiar formulation of the constitutive equations of thermoporoelasticity, i.e. involving explicitly the fluid pressure p, we will now use (85) yielding

$$\mathrm{d}g_f = \frac{\mathrm{d}p}{\rho^f} - s_f \mathrm{d}T \tag{96}$$

In addition, we will consider infinitesimal transformations allowing to replace in the state equations the Piola–Kirchhoff stress tensor $\boldsymbol{\pi}$ by the Cauchy stress tensor $\boldsymbol{\sigma}$, and the Green–Lagrange strain tensor Δ by the linearized skeleton strain tensor $\boldsymbol{\varepsilon}$ which is linked to the skeleton displacement vector $\boldsymbol{\xi}^s$ by

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\mathrm{grad}\,\boldsymbol{\xi}^s + {}^t\mathrm{grad}\,\boldsymbol{\xi}^s) \quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \xi_i^s}{\partial x_j} + \frac{\partial \xi_j^s}{\partial x_i}\right) \quad J = \varepsilon_{ii}^s = \mathrm{div}\,\boldsymbol{\xi}^s \tag{97}$$

Therefore, the state eqns (95) are rewritten in the form

$$\Psi = \Psi(\varepsilon_{ij}, m, T) \quad \sigma_{ij} = \frac{\partial \Psi}{\partial \varepsilon_{ij}} \quad g_f = \frac{\partial \Psi}{\partial m} \quad S = -\frac{\partial \Psi}{\partial T} \tag{98}$$

Differentiating the linearized eqns (98) and using (96), if follows that

$$\mathrm{d}\sigma_{ij} = \frac{\partial^2 \Psi}{\partial \varepsilon_{ij}\,\partial \varepsilon_{kl}}\mathrm{d}\varepsilon_{kl} + \frac{\partial^2 \Psi}{\partial \varepsilon_{ij}\,\partial m}\mathrm{d}m + \frac{\partial^2 \Psi}{\partial \varepsilon_{ij}\,\partial T}\mathrm{d}T \tag{99}$$

$$\frac{\mathrm{d}p}{\rho^f} = \frac{\partial^2 \Psi}{\partial \varepsilon_{ij}\,\partial m}\mathrm{d}\varepsilon_{ij} + \frac{\partial^2 \Psi}{\partial m^2}\mathrm{d}m + \left(\frac{\partial^2 \Psi}{\partial m\,\partial T} + s_f\right)\mathrm{d}T \tag{100}$$

$$\mathrm{d}S = s_f\,\mathrm{d}m - \frac{\partial^2 \Psi}{\partial T\,\partial \varepsilon_{ij}}\mathrm{d}\varepsilon_{ij} - \left(\frac{\partial^2 \Psi}{\partial m\,\partial T} + s_f\right)\mathrm{d}m - \frac{\partial^2 \Psi}{\partial T^2} \tag{101}$$

Noting

$$\frac{\partial^2\Psi}{\partial\varepsilon_{ij}\,\partial\varepsilon_{kl}} = C_{ijkl} \quad (\rho^f)^2\frac{\partial^2\Psi}{\partial m^2} = M \quad -\frac{\partial^2\Psi}{\partial T^2} = \frac{C_\varepsilon}{T}$$

$$\rho^f\frac{\partial^2\Psi}{\partial\varepsilon_{ij}\,\partial m} = -MB_{ij} \quad \rho^f\left(\frac{\partial^2\Psi}{\partial m\,\partial T}+s_f\right) = 3M\alpha_m \quad -\frac{\partial^2\Psi}{\partial T\,\partial\varepsilon_{ij}} = A_{ij} \tag{102}$$

the above incremental equations can be rewritten in the form

$$\mathrm{d}\sigma_{ij} = C_{ijkl}\,\mathrm{d}\varepsilon_{kl} - MB_{ij}\frac{\mathrm{d}m}{\rho^f} - A_{ij}\,\mathrm{d}T \tag{103}$$

$$\mathrm{d}p = M\left(-B_{ij}\,\mathrm{d}\varepsilon_{ij} + \frac{\mathrm{d}m}{\rho^f} + 3\alpha_m\,\mathrm{d}T\right) \tag{104}$$

$$\mathrm{d}S = s_f\,\mathrm{d}m + A_{ij}\,\mathrm{d}\varepsilon_{ij} - 3M\alpha_m\frac{\mathrm{d}m}{\rho^f} + \frac{C_\varepsilon}{T}\mathrm{d}T \tag{105}$$

Assume now that the tangent thermoporoelastic properties C_{ijkl}, B_{ij}, M, A_{ij} and α_m can be taken constant in a first approximation, and that the density ρ^f, the entropy s_f of the fluid and the temperature T undergo only infinitesimal variations with respect to their initial values ρ^f_0, s^0_f and T_0. Then, in a first order approximation in integration of the above equations yields

$$\sigma_{ij} = \sigma^0_{ij} + C_{ijkl}\varepsilon_{kl} - MB_{ij}\zeta - A_{ij}(T-T_0) \tag{106}$$

$$p = p_0 + M(-B_{ij}\varepsilon_{ij} + \zeta + 3\alpha_m(T-T_0)) \tag{107}$$

$$S - \rho^f_0 s^0_f\zeta = S_0 + A_{ij}\varepsilon_{ij} - 3M\alpha_m\zeta + \frac{C_\varepsilon}{T_0}(T-T_0) \tag{108}$$

where ζ is the change in volume fluid content defined by Biot (1941) (see also Detournay and Cheng (1993))

$$\zeta = \frac{m}{\rho^f_0} \tag{109}$$

Note then that we can introduce a reduced potential $W(\varepsilon_{kl}, \zeta, T-T_0)$ defined by

$$\begin{aligned} W(\varepsilon_{kl}, m, T-T_0) = {} & \sigma^0_{ij}\varepsilon_{kl} + p_0\zeta - S_0(T-T_0) + \frac{1}{2}\varepsilon_{ij}C_{ijkl}\varepsilon_{kl} \\ & -\varepsilon_{ij}A_{ij}(T-T_0) + 3\alpha_m M\zeta(T-T_0) - \zeta MB_{ij}\varepsilon_{ij} + \frac{1}{2}M\zeta^2 - \frac{1}{2}\frac{C_\varepsilon}{T_0}(T-T_0)^2 \end{aligned} \tag{110}$$

and such that

$$\sigma_{ij} = \frac{\partial W}{\partial\varepsilon_{ij}} \quad p = \frac{\partial W}{\partial\zeta} \quad S - \rho^f_0 s^0_f\zeta = -\frac{\partial W}{\partial T} \tag{111}$$

Equations (106)–(108) correspond to the state equations of linear thermoporoelasticity. Their isothermal form and the potential W have been originally introduced by Biot (1956).

They are been extended to the nonisothermal case by McTigue (1986). Note that the existence of a macro Ψ-potential is completely general, while the existence of a W-potential, from which the fluid pressure p can be derived according to (111), hinges on linearization considerations. Finally, (106), (107) and (108) simplify in the isotropic case to

$$\sigma_{ij} = \sigma_{ij}^0 + (K - \tfrac{2}{3}G)\,\varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij} - M\zeta\delta_{ij} - 3\alpha(T - T_0)\delta_{ij} \tag{112}$$

$$p = p_0 + M(-b\varepsilon_{ii} + \zeta + 3\alpha_m(T - T_0)) \tag{113}$$

$$S - \rho_0^f s_f^0 \zeta = S_0 + 3\alpha K\varepsilon_{ii} - 3M\alpha_m\zeta + \frac{C_\varepsilon}{T_0}(T - T_0) \tag{114}$$

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